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Some exact results for generalized Turán problems. (English) Zbl 1494.05058
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The authors investigate the forbidden subgraph problem in the following general setting. Let F be a fixed graph. A graph is said to be F-free if it does not contain any copy of F as a subgraph. For graphs G, H, denote by $\mathcal{N}(H,G)$ the number of copies of H in G. For fixed graphs F, H, define $\mathrm{ex}(n,H,F) := \max\{\mathcal{N}(H,G)\}$, where the maximum is taken over all F-free graphs G on n vertices. The Turán graph $T_{k-1}(n)$ is the complete $(k-1)$-partite graph on n vertices with each part containing either $\lfloor n/(k-1)\rfloor$ or $\lceil n/(k-1)\rceil$ vertices.

A classical result due to *P. Turán* [Mat. Fiz. Lapok 48, 436–452 (1941; Zbl 0026.26903)] says, in the above language, that $\mathrm{ex}(n,K_2,K_k) = \mathcal{N}(K_2,T_{k-1}(n)$. In other words, the Turán graph $T_{k-1}(n)$ has the maximum number of edges in any n-vertex K_k-free graph. In this paper, the authors examine when the Turán graph is an extremal construction for more general F and H. For a k-chromatic graph F and a graph H that is F-free, say that H is F-Turán-good if $\mathrm{ex}(n,H,F) = \mathcal{N}(H,T_{k-1}(n))$ for all sufficiently large n. When $F = K_k$, shorten this to k-Turán-good.

In Section 2, the case when $F = K_k$ is taken up:

1) The following result gives a method for constructing new k-Turán-good graphs from a given k-Turán-good graph.

Let H be a k-Turán-good graph. Let H' be any graph constructed from H in the following way. Choose a complete subgraph of H with vertex set X, add a vertex-disjoint copy of K_{k-1} to H, and join the vertices in X to the vertices of K_{k-1} by edges arbitrarily. Then H' is k-Turán-good.

2) The following is an asymptotic result on the number of copies of the path graph P_ℓ on ℓ vertices in a K_k-free graph, and indeed in any F-free graph where F has chromatic number $\chi(F) = k$.

If F is k-chromatic with $k > 2$, then $\mathrm{ex}(n,P_\ell,F) = (1+o(1))\mathcal{N}(P_\ell,T_{k-1}(n))$.

3) The main result in this section says that all complete multipartite graphs are k-Turán-good for sufficiently large k.

For every complete multipartite graph H there is an integer k_0 such that if $k \geq k_0$, then H is k-Turán-good.

Say that an edge of a graph is color-critical if the chromatic number of the graph obtained by deleting that edge is smaller than the chromatic number of the original graph. In Section 3, the authors show that the path graph P_3 is F-Turán-good for any F with chromatic number $\chi(F) \geq 4$ and with a color-critical edge. In the rest of this section, they prove specific results for small path and cycle graphs, for various F.

In Section 3.1, the case when $F = B_2$ is taken up, where the book B_k is the graph consisting of k triangles all sharing exactly one common edge. Define $\overline{M_k}$ to be the graph obtained by removing a perfect matching from K_{2k}. Define $\overline{M_k}^+$ to be the graph obtained by adding an edge to $\overline{M_k}$. Then, the following result is proved:

Let H be a $2k$-vertex graph consisting of two vertex-disjoint copies of K_k joined together with edges arbitrarily. If $\overline{M_k}$ has the maximum number of copies of H among all $2k$-vertex $\overline{M_k}^+$-free graphs, then H is $\overline{M_k}^+$-Turán-good. In particular, $\overline{M_k}$ is $\overline{M_k}^+$-Turán-good.

By specializing to the case $k = 2$, one obtains as a corollary that P_4 and C_4 are both B_2-Turán-good.

In Section 3.2, the case when F is an odd cycle is taken up. The authors show that P_4 is C_5-Turán-good. Moreover, they prove a stability result as follows:

If G is a C_5-free graph on n vertices and G has α edges contained in triangles, then $\mathcal{N}(P_4,G) \leq \mathcal{N}(P_4,T_2(n)) - (1+o(1))\alpha n^2/12$.

As a corollary to the proof, one also obtains that for a 3-chromatic graph F, if P_{2k} is F-Turán-good, then C_{2k} is F-Turán-good. In particular, C_4 is C_5-Turán-good.

In Section 3.3, the case when $F = F_2$ is taken up, where the fan F_k is the graph consisting of k triangles all sharing exactly one common vertex. It is shown that C_4 is F_2-Turán-good.

In the final section, the authors raise the following conjectures:

i) For every pair of integers ℓ and k, the path P_ℓ is k-Turán-good.

ii) For every graph H there is an integer k_0 such that if $k \geq k_0$, then H is k-Turán-good.

The authors note that the above conjecture cannot hold for all (small) k in general. They also remark that while it can be observed in some of their examples that when a graph is k-Turán-good it is also $(k+1)$-Turán-good, they do not know whether this is true for every graph H.

iii) The path P_k and the even cycle C_{2k} are $C_{2\ell+1}$-Turán-good.

Lastly, the authors note that one can also ask when the Turán graph is the unique extremal construction for these generalized Turán-type problems. Call H to be strictly F-Turán-good when this happens. The authors remark that it is not hard to show that their results for graphs F with a color-critical edge (such as when $F = B_2$ or C_5, but not F_2) also hold for the strict version.

A minor typo: read "Conjecture" in place of "Construction" on pages 4, 7, and 13.

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MSC:

05C30 Enumeration in graph theory
05C35 Extremal problems in graph theory
05C38 Paths and cycles

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References:

[1] Alon, N.; Shikhelman, C., Many \(T\) copies in \(H\)-free graphs, J. Combin. Theory Ser. B, 121, 146-172 (2016) · Zbl 1348.05100

[2] Bollobás, B.; Győri, E., Pentagons vs triangles, Discrete Math., 308, 4332-4336 (2008) · Zbl 1152.05034

[3] Cutler, J.; Nir, J.; Radcliffe, J., Supersaturation for subgraph counts (2019), arXiv:1903.08059 · Zbl 1485.05083

[4] Erdős, P., On the number of complete subgraphs contained in certain graphs, Magyar Tud. Akad. Mat. KutatÓInt. KÖzl., 7, 459-464 (1962) · Zbl 0116.01202

[5] Erdős, P.; Füredi, Z.; Gould, R. J.; Gunderson, D. S., Extremal graphs for intersecting triangles, J. Combin. Theory Ser. B, 64, 89-100 (1995) · Zbl 0822.05036

[6] Erdős, P.; Simonovits, M., A limit theorem in graph theory, Studia Sci. Math. Hungar., 1, 51-57 (1966) · Zbl 0178.27301

[7] Erdős, P.; Stone, A. H., On the structure of linear graphs, Bull. Am. Math. Soc., 52, 1087-1091 (1946) · Zbl 0063.01277

[8] Ergemlidze, B.; Győri, E.; Methuku, A.; Salia, N., A note on the maximum number of triangles in a \(C_5\)-free graph, J. Graph Theory, 90, 227-230 (2019) · Zbl 1407.05133

[9] Ergemlidze, B.; Methuku, A., Triangles in \(C_5\)-free graphs and hypergraphs of girth six (2018), arXiv:1811.11873

[10] Gerbner, D., Generalized turán problems for small graphs (2020), arXiv:2006.16150

[11] Gerbner, D.; Győri, E.; Methuku, A.; Vizer, M., Generalized turán problems for even cycles, J. Combin. Theory Ser. B, 145, 169-213 (2019) · Zbl 1448.05107

[12] Gerbner, D.; Methuku, A.; Vizer, M., Generalized Turán problems for disjoint copies of graphs, Discrete Math., 342, 3130-3141 (2019) · Zbl 1419.05110

[13] Gerbner, D.; Palmer, C., Counting copies of a fixed subgraph in \(F\)-free graphs, European J. Combin., 82, Article 103001 pp. (2019) · Zbl 1419.05104

[14] Gishboliner, L.; Shapira, A., A generalized turán problem and its applications, Int. Math. Res. Notices (IMRN), 11, 3417-3452 (2020) · Zbl 1483.05082

[15] Grzesik, A., On the maximum number of five-cycles in a triangle-free graph, J. Combin. Theory Ser. B, 102, 1061-1066 (2012) · Zbl 1252.05093

[16] Guiduli, B., Spectral Extrema for Graphs (1996), University of Chicago, (Ph.D. thesis)

[17] Győri, E., On the number of \(C_5\)'s in a triangle-free graph, Combinatorica, 9, 101-102 (1989) · Zbl 0679.05042

[18] Győri, E.; Li, H., The maximum number of triangles in \(C_{2 k + 1}\)-free graphs, Combin. Probab. Comput., 21, 187-191

(2011) · Zbl 1238.05145

[19] Győri, E.; Pach, J.; Simonovits, M., On the maximal number of certain subgraphs in \(K_r\)-free graphs, Graphs Combin., 7, 31-37 (1991) · Zbl 0755.05050

[20] Győri, E.; Salia, N.; Tompkins, C.; Zamora, O., The maximum number of \(P_\ell\) copies in \(P_k\)-free graphs, Discrete Math. Theor. Comput. Sci., 21, 21 (2019), Paper No. 14 · Zbl 1417.05106

[21] Hatami, H.; Hladký, J.; Král', D.; Norine, D.; Razborov, A., On the number of pentagons in triangle-free graphs, J. Combin. Theory Ser. A, 120, 722-732 (2012) · Zbl 1259.05087

[22] Ma, Jie; Qiu, Yu, Some sharp results on the generalized Turán numbers, European J. Combin., 84, Article 103026 pp. (2020) · Zbl 1428.05141

[23] Nikiforov, V., A spectral erdős-Stone-Bollobás theorem, Combin. Probab. Comput., 18, 455-458 (2009) · Zbl 1208.05077

[24] Pippenger, N.; Golumbic, M. C., The inducibility of graphs, J. Comb. Theory Ser. B, 19, 189-203 (1975) · Zbl 0332.05119

[25] Simonovits, M., A method for solving extremal problems in graph theory, stability problems, (Theory of Graphs, Proc. Colloq., Tihany (1966), Academic Press: Academic Press New York), 279-319 · Zbl 0164.24604

[26] Turán, P., On an extremal problem in graph theory (in hungarian), Mat. Fizikai Lapok, 48, 436-452 (1941) · Zbl 67.0732.03

[27] Zykov, A. A., On some properties of linear complexes, Mat. Sb., 66, 163-188 (1949) · Zbl 0033.02602